Robotics

Jacobians: Velocities and Static Forces

INTRODUCTION

- In this chapter ,we expand our consideration of robot manipulators beyond static-positioning Problems.

- We examine the notions of linear and angular velocity of a rigid body and use these Concepts to analyze the motion of a manipulator.

5.7 Jacobians

$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$\vdots$$

$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6).$$

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6,$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6,$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6,$$

$$\delta Y = \frac{\partial F}{\partial X}\delta X.$$

$$\delta Y = J(X)\delta X.$$

$$\dot{Y} = J(X)\dot{X}.$$

$${}^{0}\nu = {}^{0}J(\Theta)\dot{\Theta},$$

Changing a Jacobian's frame of reference

$$\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}_{6\times 1} = {}^{B}\nu = {}^{B}J(\Theta)_{6\times n}\dot{\Theta}_{n\times 1},$$

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & {}^{A}_{B}R_{3\times 3} \end{bmatrix} \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}.$$

$${}^{A}J_{6\times n}(\Theta) = \begin{bmatrix} {}^{A}_{B}R_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & {}^{A}_{B}R_{3\times 3} \end{bmatrix} {}^{B}J_{6\times n}(\Theta).$$

Example

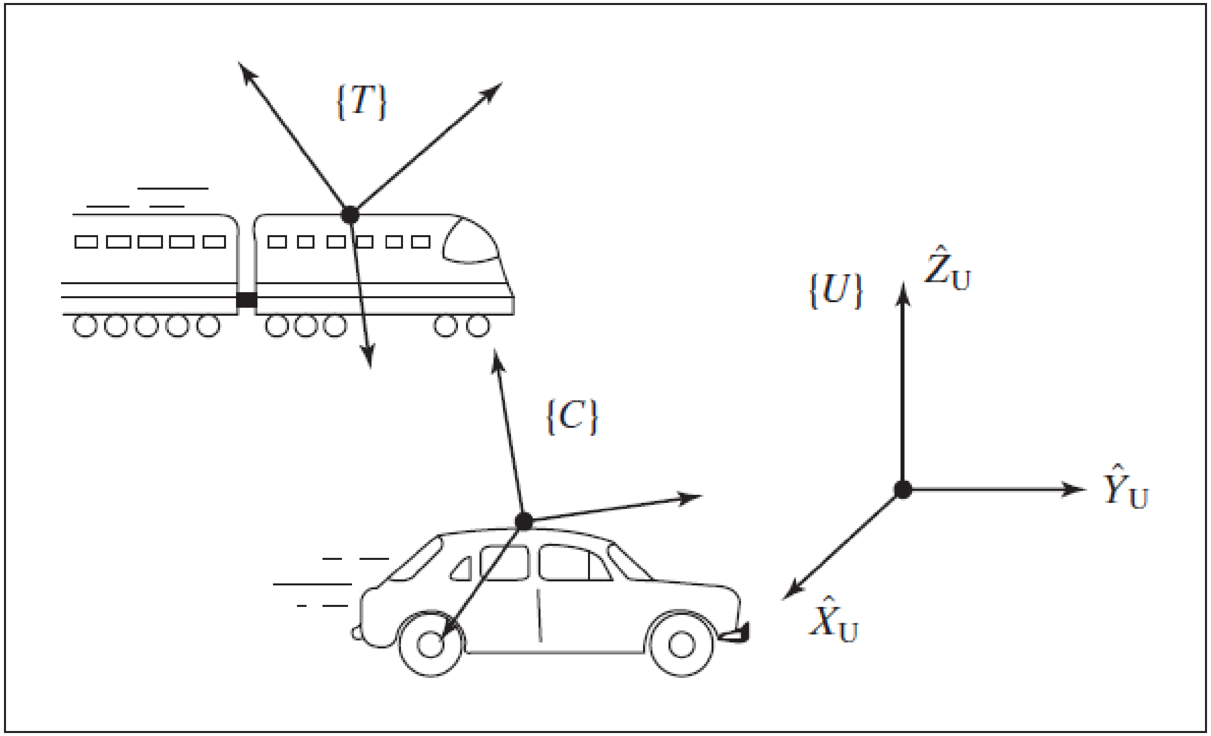


FIGURE 5.1: Example of some frames in linear motion.

Example

EXAMPLE 5.1

Figure 5.1 shows a fixed universe frame, $\{U\}$, a frame attached to a train traveling at 100 mph, $\{T\}$, and a frame attached to a car traveling at 30 mph, $\{C\}$. Both vehicles are heading in the $\hat{X}$ direction of $\{U\}$. The rotation matrices, ${}^U_T R$ and ${}^U_C R$, are known and constant.

What is $\frac{{}^U d}{dt}\,{}^U P_{CORG}$?

$$\frac{{}^U d}{dt}\,{}^U P_{CORG} = {}^U V_{CORG} = \upsilon_C = 30\hat{X}.$$

What is ${}^C({}^U V_{TORG})$?

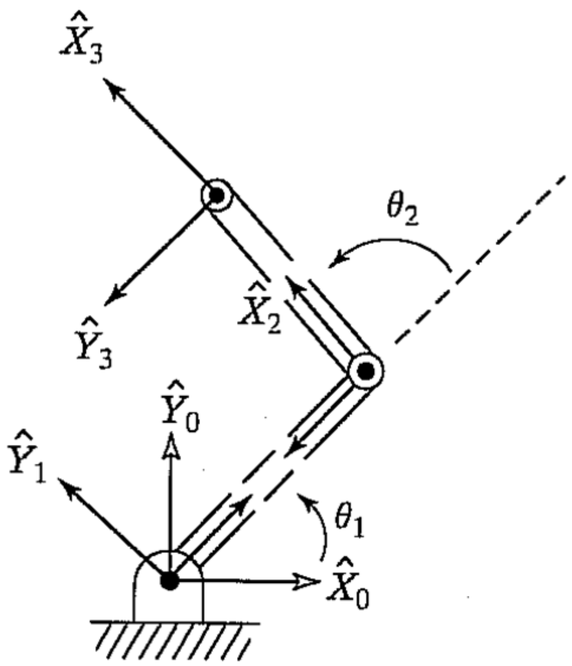
$${}^C({}^U V_{TORG}) = {}^C\upsilon_T = {}^C_U R\,\upsilon_T = {}^C_U R(100\hat{X}) = {}^U_C R^{-1}\,100\hat{X}.$$

What is ${}^C({}^T V_{CORG})$?

$${}^C({}^T V_{CORG}) = {}^C_T R\;{}^T V_{CORG} = -{}^U_C R^{-1}\;{}^U_T R\;70\hat{X}.$$

EXAMPLE 1

A two-link manipulator with rotational joints is shown below, Calculate the velocity of the tip of the arm (end effector) as a function of joint rates (joint velocity). Give the answer in two forms- in terms of frame {3} ${}^{3}_{3}V$ and also in terms of frame {0} ${}^{0}_{3}V$.



$${}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^{1}_{2}T = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- See notes

5.7 Jacobians

v : Cartesian velocity = End-effector velocity.

$$v = \begin{bmatrix} \text{linear velocity} \\ \text{angular velocity} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix} = J \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \vdots \\ \dot\theta_n \end{bmatrix}$$

$$P_x(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial P_x}{\partial t} = v_x = \frac{\partial P_x}{\partial \theta_1}\dot\theta_1 + \frac{\partial P_x}{\partial \theta_2}\dot\theta_2 + \cdots \frac{\partial P_x}{\partial \theta_n}\dot\theta_n$$

$$P_y(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial P_y}{\partial t} = v_y = \frac{\partial P_y}{\partial \theta_1}\dot\theta_1 + \frac{\partial P_y}{\partial \theta_2}\dot\theta_2 + \cdots \frac{\partial P_y}{\partial \theta_n}\dot\theta_n$$

$$P_z(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial P_z}{\partial t} = v_z = \frac{\partial P_z}{\partial \theta_1}\dot\theta_1 + \frac{\partial P_z}{\partial \theta_2}\dot\theta_2 + \cdots \frac{\partial P_z}{\partial \theta_n}\dot\theta_n$$

$$\phi_x(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial \phi_x}{\partial t} = w_x = \frac{\partial \phi_x}{\partial \theta_1}\dot\theta_1 + \cdots \frac{\partial \phi_x}{\partial \theta_n}\dot\theta_n$$

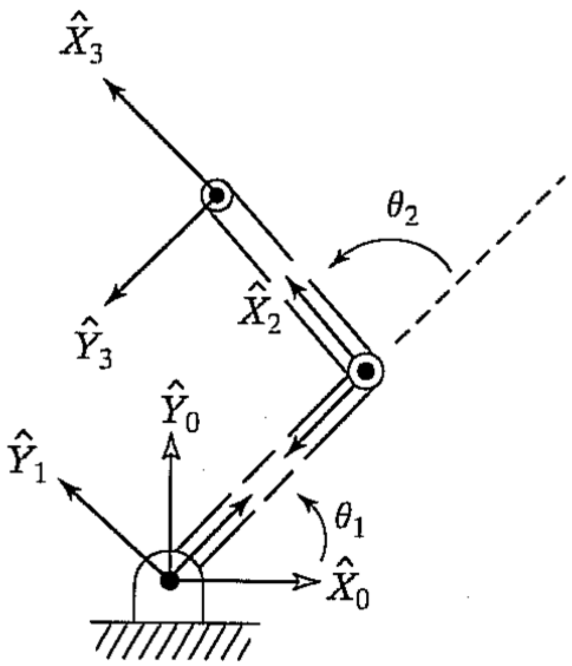
$$\phi_y(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial \phi_y}{\partial t} = w_y = \frac{\partial \phi_y}{\partial \theta_1}\dot\theta_1 + \cdots + \frac{\partial \phi_y}{\partial \theta_n}\dot\theta_n$$

$$\phi_z(\theta_1, \theta_2 \ldots \theta_n) \rightarrow \frac{\partial \phi_z}{\partial t} = w_z = \frac{\partial \phi_z}{\partial \theta_1}\dot\theta_1 + \cdots + \frac{\partial \phi_z}{\partial \theta_n}\dot\theta_n$$

$$\begin{matrix} v \left\{ \right. \\ \\ w \left\{ \right. \end{matrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \underbrace{\begin{bmatrix} \partial P_x/\partial\theta_1 & \partial P_x/\partial\theta_2 & \cdots & \partial P_x/\partial\theta_n \\ \partial P_y/\partial\theta_1 & \partial P_y/\partial\theta_2 & \cdots & \partial P_y/\partial\theta_n \\ \partial P_z/\partial\theta_1 & \partial P_z/\partial\theta_2 & \cdots & \partial P_z/\partial\theta_n \\ \partial \phi_x/\partial\theta_1 & \partial \phi_x/\partial\theta_2 & \cdots & \partial \phi_x/\partial\theta_n \\ \partial \phi_y/\partial\theta_1 & \partial \phi_y/\partial\theta_2 & \cdots & \partial \phi_y/\partial\theta_n \\ \partial \phi_z/\partial\theta_1 & \partial \phi_z/\partial\theta_3 & \cdots & \partial \phi_z/\partial\theta_n \end{bmatrix}}_{6 \times n} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \vdots \\ \dot\theta_n \end{bmatrix}$$

EXAMPLE 1 (another method)

A two-link manipulator with rotational joints is shown below, Calculate the velocity of the tip of the arm (end effector) as a function of joint rates (joint velocity). Give the answer in two forms- in terms of frame {3} ${}^3_3V$ and also in terms of frame {0} ${}^0_3V$.



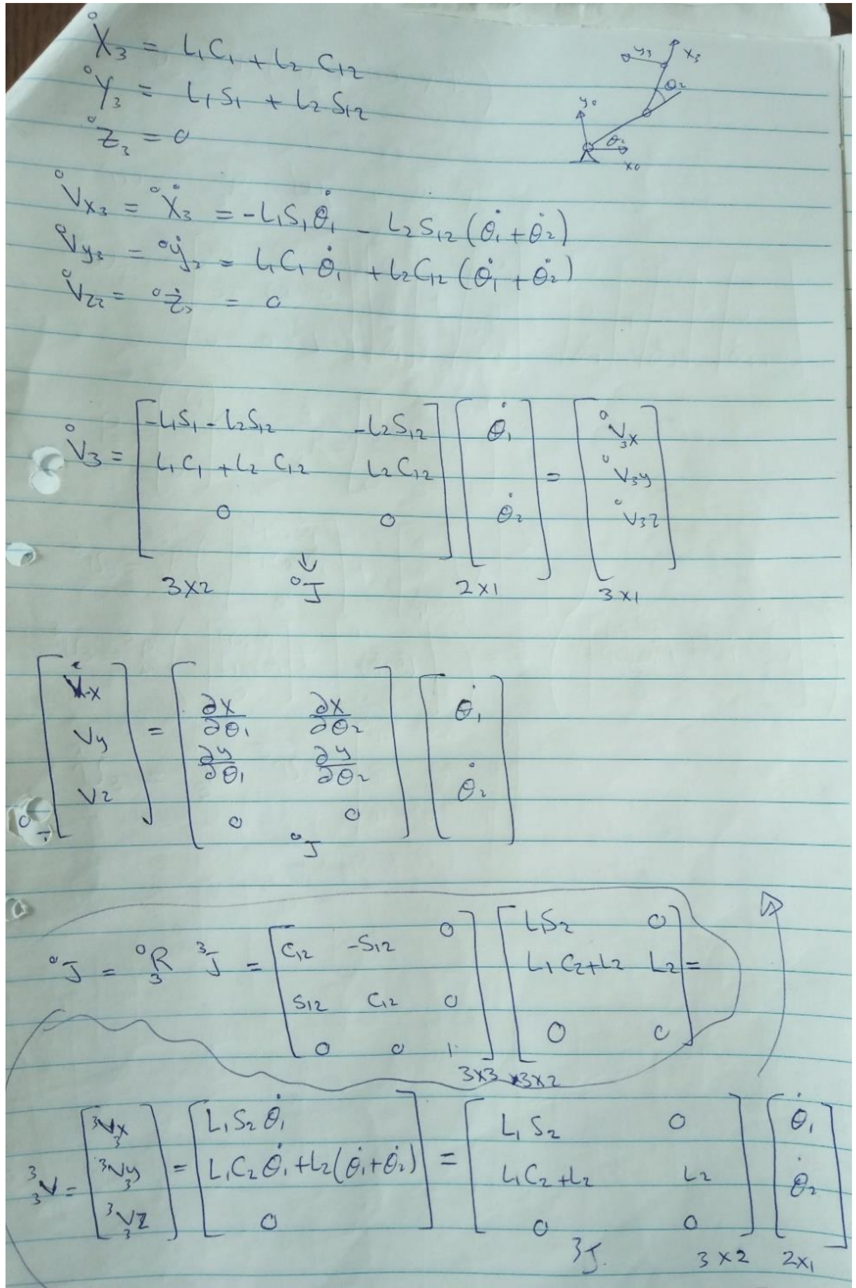
$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^1_2T = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

- See notes

• Example 1 (another method)



$^{0}X_3 = L_1C_1 + L_2C_{12}$

$^{0}Y_3 = L_1S_1 + L_2S_{12}$

$^{0}Z_3 = 0$

$^{0}V_{x_3} = {}^{0}\dot{X}_3 = -L_1S_1\dot{\theta}_1 - L_2S_{12}(\dot{\theta}_1+\dot{\theta}_2)$

$^{0}V_{y_3} = {}^{0}\dot{y}_3 = L_1C_1\dot{\theta}_1 + L_2C_{12}(\dot{\theta}_1+\dot{\theta}_2)$

$^{0}V_{z_3} = {}^{0}\dot{z}_3 = 0$

$$^{0}V_3 = \underbrace{\begin{bmatrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \\ 0 & 0 \end{bmatrix}}_{3\times2,\ ^{0}J} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} ^{0}V_{3x} \\ ^{0}V_{3y} \\ ^{0}V_{3z} \end{bmatrix}}_{3\times1}$$

$$^{0}\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \\ 0 & 0 \end{bmatrix}}_{^{0}J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^{0}J = {}^{0}_{3}R\ {}^{3}J = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1S_2 & 0 \\ L_1C_2+L_2 & L_2 \\ 0 & 0 \end{bmatrix} =$$

$3\times3 \times 3\times2$

$$^{3}_{3}V = \begin{bmatrix} ^{3}V_x \\ ^{3}V_y \\ ^{3}V_z \end{bmatrix} = \begin{bmatrix} L_1S_2\dot{\theta}_1 \\ L_1C_2\dot{\theta}_1 + L_2(\dot{\theta}_1+\dot{\theta}_2) \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} L_1S_2 & 0 \\ L_1C_2+L_2 & L_2 \\ 0 & 0 \end{bmatrix}}_{^{3}J,\ 3\times2} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{2\times1}$$

- Example 1 (another method)

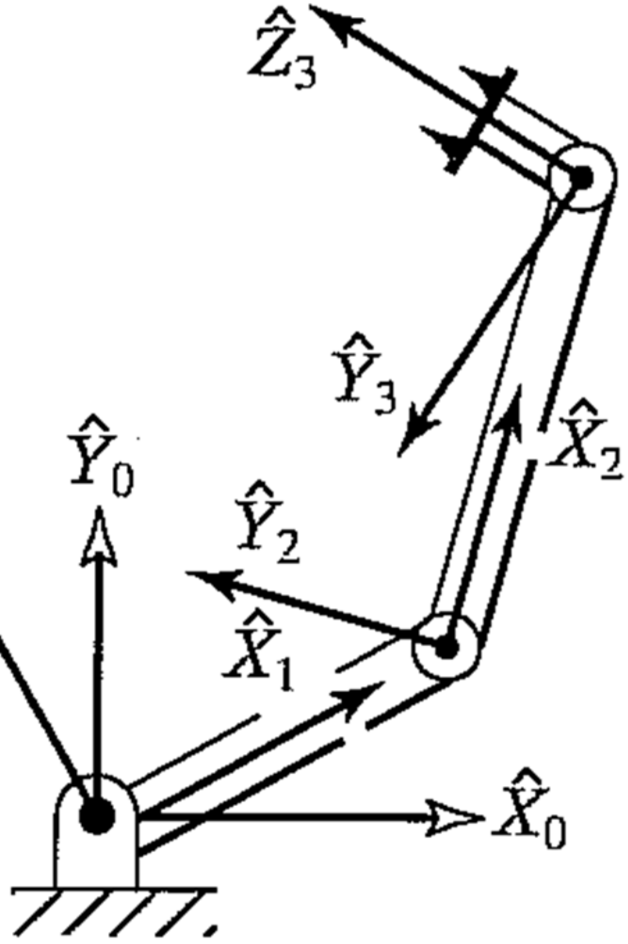
$$ {}^{0}J = \begin{bmatrix} C_{12} L_1 S_2 - S_{12} L_1 C_2 - S_{12} L_2 & -S_{12} L_2 \\ S_{12} L_1 S_2 + L_1 C_2 C_{12} + C_{12} L_2 & C_{12} L_2 \\ 0 & 0 \end{bmatrix}_{3\times 2} $$

$$ {}^{0}J = \begin{bmatrix} L_1 [S(\theta_2 - (\theta_1 + \theta_2))] - S_{12} L_2 & -L_2 S_{12} \\ L_1 [C(\theta_2 - \theta_1 - \theta_2)] + C_{12} L_2 & L_2 C_{12} \\ 0 & 0 \end{bmatrix} $$

$$ {}^{0}J = \begin{bmatrix} L_1 S(-\theta_1) - S_{12} L_2 & -L_2 S_{12} \\ L_1 C(-\theta_1) + C_{12} L_2 & L_2 C_{12} \\ 0 & 0 \end{bmatrix} $$

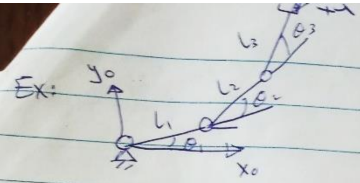
$$ {}^{0}J = \begin{bmatrix} -L_1 S_1 - S_{12} L_2 & -L_2 S_{12} \\ L_1 C_1 + C_{12} L_2 & L_2 C_{12} \\ 0 & 0 \end{bmatrix} $$

- Example 3



- Example 3

Ex:



$$ {}^0P_{4x} = L_1C_1 + L_2C_{12} + L_3C_{123} $$
$$ {}^0P_{4y} = L_1S_1 + L_2S_{12} + L_3S_{123} $$
$$ {}^0P_{4z} = 0 $$
$$ \phi_x = 0 $$
$$ \phi_y = 0 $$
$$ \phi_z = \theta_1 + \theta_2 + \theta_3 $$

$$ {}^0J = \begin{bmatrix} \frac{\partial P_x}{\partial \theta_1} & \frac{\partial P_x}{\partial \theta_2} & \frac{\partial P_x}{\partial \theta_3} \\ \vdots & & \\ \frac{\partial \phi_z}{\partial \theta_1} & \frac{\partial \phi_z}{\partial \theta_2} & \frac{\partial \phi_z}{\partial \theta_3} \end{bmatrix} $$

$$ {}^0J = \begin{bmatrix} -L_1S_1 - L_2S_{12} - L_3S_{123} & -L_2S_{12} - L_3S_{123} & -L_3S_{123} \\ L_1C_1 + L_2C_{12} + L_3C_{123} & L_2C_{12} + L_3C_{123} & L_3C_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} $$

6×3

OR

$$ {}^0J = \begin{bmatrix} -L_1S_1 - L_2S_{12} - L_3S_{123} & -L_2S_{12} - L_3S_{123} & -L_3S_{123} \\ L_1C_1 + L_2C_{12} + L_3C_{123} & L_2C_{12} + L_3C_{123} & L_3C_{123} \\ 1 & 1 & 1 \end{bmatrix} $$

3×3

Planar

(8)

Changing a Jacobian's frame of reference